Journal of Statistical Physics, Vol. 28, No. 2, 1982

The Asymptotic Behavior of a Random Walk on a Dual-Medium Lattice

C. C. Heyde,[1] M. Westcott,[1] and E. R. Williams[1]

Received August 4, 1981; revised November 10, 1981

This paper considers the asymptotic distribution for the horizontal displacement of a random walk in a medium represented by a two-dimensional lattice, whose transitions are to nearest-neighbor sites, are symmetric in the horizontal and vertical directions, and depend on the column currently occupied. On either side of a change-point in the medium, the transition probabilities are assumed to obey an asymptotic density condition. The displacement, when suitably normalized, converges to a diffusion process of oscillating Brownian motion type. Various special cases are discussed.

KEY WORDS: Random walk; anisotropic lattice; change-point; diffusion limit; oscillating Brownian motion.

1. INTRODUCTION

Recently, there has been interest in properties of random walks on several classes of anisotropic lattices which have different scattering characteristics for the walk from the different types of lattice sites.[1–3] A theme of this work is that the asymptotic properties of the random walk, and hence of the anisotropic diffusion it is supposed to approximate, depend only on the relative densities of the different types of scatterers and not on their spatial arrangements; specific instances of this result date back to Maxwell and Rayleigh. This paper will demonstrate that such results are to be expected for a much wider class of properties than previously considered.

These walks serve as models for transport through an anisotropic medium. The case discussed in Refs. 1 and 2 is that of a two-dimensional lattice with two types of column, called "strong" and "weak," having different scattering characteristics. Our lattices are a multitype generalization of this arrangement.[4,5]

[1] CSIRO Division of Mathematics and Statistics, Canberra, A.C.T.

0022-4715/82/0600-0375$03.00/0

Seshadri et al.[2] successfully analyzed properties of their walk such as mean-square displacements of the horizontal and vertical components, the probability of return to the origin, and the range, in the case of a strictly periodic array of the two-column types. Their proofs for more general arrays appear to have some technical deficiencies. Westcott,[4] using a probabilistic argument, generalized the mean-square displacement results to an arbitrary collection of column types satisfying a uniform asymptotic density condition. His method provides an elementary treatment of the periodic case with arbitrary types. By much more subtle techniques, it is possible to show that the horizontal displacement obeys a central limit theorem without the requirement of uniformity in the asymptotic density; this was pointed out by a referee of Ref. 4. Finally, Heyde[5] has established the asymptotic equivalence of the horizontal motion of the walk to rescaled Brownian motion, under an overlapping but somewhat different assumption to that of Ref. 4. This suggests that diffusion-limiting results may hold more generally, and that some asymptotic density condition is the fundamental requirement.

Thinking of the two-column type case again, all these results contain some version of the assumption that the strong columns, say, occur in a roughly constant proportion *throughout the entire medium*. A simple example is when strong and weak columns strictly alternate. A more dramatic form of anisotropy is to assume there is an abrupt change in the nature of the diffusion medium at some point, when one expects significant changes in the behavior of the process on each side of this point; for example, when all columns to the left of some point are strong while those to the right are weak. In both these examples there is, in some sense, a 1 : 1 "mix" of strong and weak columns, and a question of considerable interest is whether the asymptotic results mentioned above, which certainly hold for the former, continue to hold for the latter.

This question cannot apparently be resolved using the methods of previous work, since all rely critically at some stage on the asymptotic column-type densities on either side of any point existing *and being equal*. However, a general approach which covers all the cases is provided by work of Stone[6] in which probabilistic methods, based on the use of local time for Brownian motion, are used to establish convergence of birth and death processes and random walks to limiting diffusions. A crucial requirement is that the processes are "skip-free" in both directions, but this is the case in the present context. More analytical procedures seem necessary to deal with walks which allow more general steps than 0, $+1$, or -1; for deep results on their recurrence properties (which do not concern us here) see Kemperman.[7]

We note here that Stone's methods actually prove convergence of the random walk, and some associated quantities, in a rather stronger sense than the convergence in distribution usually associated with central-limit-type results. This will be discussed further in Section 4.

2. MODEL AND RESULTS

We consider a random walk on a unit square lattice in $\mathbb{R}^2$ which, from a lattice site on column j, moves to either horizontal neighbor with probability p_j and to either vertical neighbor with probability $\frac{1}{2} - p_j$. We concentrate on the horizontal motion, described more formally as follows.

Let $\{X_n\}$ $(n = 0, 1, \ldots)$, $X_0 = 0$, be a symmetric random walk on the integers with transition probabilities

$$\begin{aligned} P(X_{n+1} = j+1 \mid X_n = j) &= p_j \\ P(X_{n+1} = j \mid X_n = j) &= 1 - 2p_j \end{aligned} \tag{1}$$

for $j \in \mathbb{Z}$ and $n = 0, 1, \ldots$. To avoid triviality, take $p_j > 0$ $(j \in \mathbb{Z})$.

Let

$$\frac{1}{k} \sum_{j=-k}^{-1} \frac{1}{p_j} = \gamma' + \epsilon'_k, \qquad \frac{1}{k} \sum_{j=1}^{k} \frac{1}{p_j} = \gamma + \epsilon_k \tag{2}$$

The *periodic case* occurs when, for some positive integers M, M',

$$p_j = p_{j-M'} \quad (j \leqslant 0), \qquad p_j = p_{j+M} \quad (j \geqslant 0)$$

Finally, we use $\xrightarrow{\text{a.s.}}$ and $\xrightarrow{d}$ to denote convergence almost surely and in distribution, respectively, and $[x]$ to denote the integer part of x.

Theorem. Suppose that ϵ'_k and ϵ_k are $o(1)$ as $k \to \infty$. Then,

$$\sup_{0 \leqslant t \leqslant N} |n^{-1/2} X_{[nt]} - Y(t)| \xrightarrow{\text{a.s.}} 0 \tag{3}$$

as $n \to \infty$ for all $N > 0$ where $\{Y(t), t \geqslant 0\}$ is a diffusion process on the same probability space as $\{X_n\}$ whose distribution is defined by

$$Y(t) = W\big(A^{-1}(t)\big), \qquad t \geqslant 0$$

$\{W(t), t \geqslant 0\}$ being standard Brownian motion and

$$A(t) = \int_0^t \sigma^{-2}(W(s))\, ds$$

where $\sigma^2(y) = 2/\gamma, y \geqslant 0, = 2/\gamma', y < 0$.

The process $Y(t)$ of the theorem, called *oscillating Brownian motion* if $\gamma \neq \gamma'$, is the diffusion with speed measure $m(dy) = 2\sigma^{-2}(y)\, dy$, $\sigma^2(y)$ being defined above.

The theorem substantially extends the previous work on the subject except for that of Heyde[5] where a stronger form of convergence result is obtained but in a more restricted setting and under less general conditions.

Corollary. Under the assumptions of the theorem,

(i)
$$n^{-1/2}X_n \xrightarrow{d} Y(1)$$

as $n \to \infty$ where $Y(1)$ has density given by

$$f(y) = \begin{cases} (\gamma'/\pi)^{1/2}(1-\theta)e^{-\gamma' y^2/4}, & y < 0 \\ (\gamma/\pi)^{1/2}\theta e^{-\gamma y^2/4}, & y \geq 0 \end{cases}$$

where $\theta = \sqrt{\gamma}/(\sqrt{\gamma} + \sqrt{\gamma'})$.

(ii)
$$n^{-1}\langle X_n^2 \rangle \to 2/\sqrt{\gamma\gamma'}$$

as $n \to \infty$.

(iii) The theorem, and (i), (ii) above, continue to hold if the p_j are random variables independent of the process, ϵ_k and ϵ_k' are $o(1)$ a.s., and we take expectations over the p_j.

Note that, in the case $\gamma = \gamma'$, (ii) is the result of Ref. 4 without any uniformity assumption while (i) is the result proved by the referee of Ref. 4 by a different method.

3. PROOFS

Define the normalized process $W_n(t) = n^{-1/2}X_{[nt]}$, $t \geq 0$. Then, in the notation of Stone we have for $n \geq 1$, $E_n = \{i/n^{1/2},\ -\infty < i < \infty\}$, $\theta_n = 1/n$, $x_n = 0$, $q_i^{(n)} = 0$, $\alpha_i^{(n)} = i/n^{1/2}$, and

$$P\big(W_n((k+1)/n) = (i \pm 1)/n^{1/2} \mid W_n(k/n) = i/n^{1/2}\big) = p_i$$
$$P\big(W_n(k/n) = i/n^{1/2} \mid W_n(k/n) = i/n^{1/2}\big) = 1 - 2p_i$$

while

$$m_n\big(\{i/n^{1/2}\}\big) = 1/\big(p_i n^{1/2}\big)$$

so that

$$\lim_{n\to\infty} m_n(x) = m(x) = \begin{cases} \gamma x, & x \geq 0 \\ \gamma' x, & x < 0 \end{cases}$$

using Eq. (2). Then, $m(x)$ is just the speed measure of an oscillating Brownian motion $Y(t)$, say, with $\sigma_+^2 = 2/\gamma$, $\sigma_-^2 = 2/\gamma'$ (Keilson and

Wellner[8]). The process $Y(t)$ is continuous a.s. and Theorem 2 of Stone[6] immediately gives the result of the theorem.

Since W_n converges weakly to the diffusion process Y we have

$$W_n(1) = n^{-1/2}X_n \xrightarrow{d} Y(1)$$

and the form for the density of $Y(1)$ is given in Corollary 1 of Keilson and Wellner.[8] This deals with part (i) of the Corollary.

Part (ii) of the Corollary is obtained by noting that $\{n^{-1}X_n^2, n \geqslant 1\}$ is uniformly integrable using the argument in the proof of Corollary 2 of Heyde.[5] The required result then follows similarly since $\langle Y^2(1)\rangle = 2/\sqrt{\gamma\gamma'}$. Part (iii) is a consequence of dominated convergence.

4. AN EXAMPLE AND DISCUSSION

A striking illustration of our results comes from considering the case[1,2] of strong and weak columns in a periodic setting. Let t, t', respectively, be the transition probabilities of strong and weak columns; choose $M = M'$ and take $p_j = t'$ $(j < 0,\ j \neq -rM)$, $p_j = t$ $(j = -rM)$, $p_j = t$ $(j > 0, j \neq rM)$, $p_j = t'$ $(j = rM)$, where $r = 1, 2, \ldots$. That is, to the left of the origin every Mth column is strong, the others being weak, while to the right the situation is reversed. Then a little algebra shows that

$$\mathrm{Var}(X_n) \sim 2n/\left\{\beta G^2 + (1-\beta)A^2\right\}^{1/2} \tag{4}$$

where

$$G = \left(\frac{1}{tt'}\right)^{1/2}, \qquad A = \frac{1}{2}\left(\frac{1}{t} + \frac{1}{t'}\right), \qquad \beta = \left(1 - \frac{2}{M}\right)^2$$

That is, the diffusion coefficient involves a mixture of the arithmetic and geometric means of $1/t$ and $1/t'$. When $M = 2$, the medium is regularly anisotropic (alternating strong and weak columns) and we have A; when $M = 1$ or ∞ there are two semi-infinite blocks, of strong and weak columns, respectively, the medium has an abrupt change in anisotropy, and we have G. (These are the two examples of Section 1.) Equation (4), then, provides a natural occurrence of a smooth transition between A and G as the irregularity of the medium's anisotropy increases.

The choice of zero as both the starting point of $\{X_n\}$ and the change point in the medium is purely for convenience. It is clear that the results still hold if these two points differ, since only a finite number of steps is involved and the walk is null-recurrent whenever (i) holds.

Part (ii) of the Corollary provides the vertical mean-square displacement also since, by symmetry, the mean-square displacement after n steps for the two-dimensional walk is n. The distribution of the vertical displacement is governed by a rather different mechanism and will not be treated here. Results about the two-dimensional walk will also require another approach, since Stone's method is closely tied to one-dimensional processes.

Finally, we say a little about the type of convergence proved in the theorem, and its consequences. Equation (3) states that the path of our random walk, suitably normalized, is uniformly close to the path of an oscillating Brownian motion over any finite time interval, with probability one. This is clearly a very strong result, considerably more far-reaching than the classical central limit type of convergence deduced as (i). One important immediate consequence of Eq. (3) is that the normalized walk converges weakly to oscillating Brownian motion, meaning that the probability measures on the space of paths converge, if we endow the space with a suitable topology. And then we can deduce that any functional of the paths which is continuous in this topology almost surely with respect to oscillating Brownian motion converges to the same functional of oscillating Brownian motion (Billingsley,[9] Theorem 5.1). So there is a very extensive set of asymptotic properties of the walk which depend only on asymptotic densities existing; examples are the supremum of the process and the occupation time of particular sets. This is the basis of our assertion in the Introduction that the property of dependence only on density is ubiquitous for such walks.

REFERENCES

1. K. E. Shuler, *Physica*, **95A**:12 (1979).
2. V. Seshadri, K. Lindenberg, and K. E. Shuler, *J. Stat. Phys.* **21**:517 (1979).
3. K. E. Shuler and U. Mohanty, *Proc. Nat. Acad. Sci.* to appear (1981).
4. M. Westcott, *J. Stat. Phys.* **27**:75 (1982).
5. C. C. Heyde, *J. Stat. Phys.* **27**:721 (1982).
6. C. Stone, *Illinois J. Math.* **7**:638 (1963).
7. J. B. Kemperman, *Stoc. Proc. Appl.* **2**:1 (1974).
8. J. Keilson and J. A. Wellner, *J. Appl. Prob.* **15**:300 (1978).
9. P. Billingsley, *Convergence of Probability Measures* (Wiley, 1968).